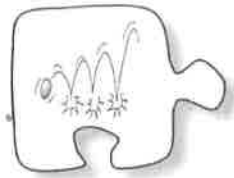
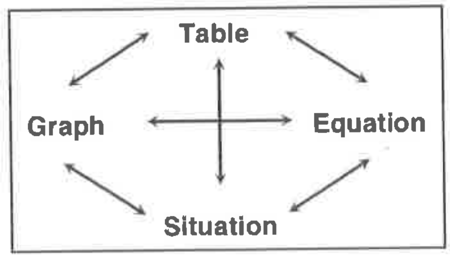
5.1.1 How does the pattern grow?

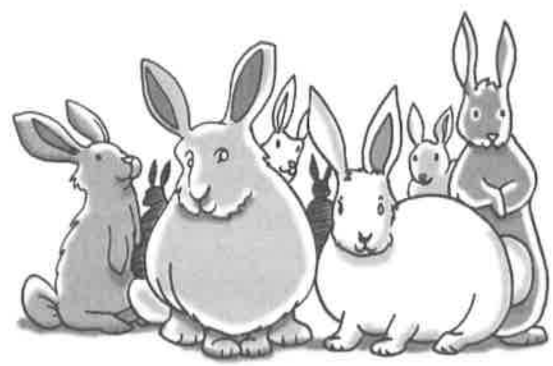


Representing Exponential Growth



So far in this course, you have been investigating the family of linear functions using multiple representations (especially tables, graphs, and equations). In this chapter, you will learn about a new family of functions and the type of growth it models.

5-1. MULTIPLYING LIKE BUNNIES



In the book *Of Mice and Men* by John Steinbeck, two good friends named Lenny and George dream of raising rabbits and living off the land. What if their dream came true?

Suppose Lenny and George started with two rabbits and that in each month following those rabbits have two babies. Also suppose that every month thereafter, each pair of rabbits has two babies.

Your Task: With your team, determine how many rabbits Lenny and George would have after one year (12 months). Represent this situation with a written description of the pattern of growth, a diagram, and a table. What patterns can you find and how do they compare to other patterns that you have investigated previously? **[They would have 8,192 rabbits after one year.]**

Discussion Points

What strategies could help us keep track of the total number of rabbits?

What patterns can we see in the growth of the rabbit population?

How can we predict the total number of rabbits after many months have passed?

Further Guidance

5-2. How can you determine the number of rabbits that will exist at the end of one year? Consider this as you answer the questions below.

a. Draw a diagram to represent how the total number of rabbits is growing each month. How many rabbits will Lenny and George have after three months? **[Diagrams vary. At the end of one month, there are four rabbits (the original two and their two offspring), so there will be 16 rabbits after three months.]**

b. As the number of rabbits becomes larger, a diagram becomes too cumbersome to be useful. A table might work better. Organize your information in a table showing the total number of rabbits for the first several months (at least 6 months). What patterns can you find in your table? Describe the pattern of growth in words. **[The number of rabbits begins with 2 and doubles every month.]**

c. If you have not done so already, use your pattern to determine the number of rabbits that Lenny and George would have after one year (12 months) have passed. **[At the end of 12 months there will be 8192 rabbits.]**

d. How does the growth in the table that you created compare to the growth patterns that you have investigated previously? How is it similar and how is it different? **[Students will likely notice that this growth pattern is not linear or that it does not grow by a constant amount.]**

Further Guidance section ends here.

5-3. Lenny and George want to raise as many rabbits as possible, so they have a few options to consider. They could start with a larger number of rabbits, or they could raise a breed of rabbits that reproduces faster. How do you think that each of these options would change the pattern of growth you observed in the previous problem? Which situation might yield the largest rabbit population after one year?

a. To help answer these questions, model each case below with a table for the first five months. **[See tables below.]**

Case 2: Start with 10 rabbits; each pair has 2 babies per month.

Case 3: Start with 2 rabbits; each pair has 4 babies per month.

Case 4: Start with 2 rabbits; each pair has 6 babies per month.

b. Which case would appear to give Lenny and George the most rabbits after one year? How many rabbits would they have in that case? **[Case 4 appears to result in the most rabbits at the end of one year; they would have 33,554,432 rabbits.]**

Case 2	
Months	Rabbits
0	10
1	20
2	40
3	80
4	160
5	320
6	640
7	1,280
8	2,560
9	5,120
10	10,240
11	20,480
12	40,960

Case 3	
Months	Rabbits
0	2
1	6
2	18
3	54
4	162
5	486
6	1,458
7	4,374
8	13,122
9	39,366
10	118,098
11	354,294
12	1,062,882

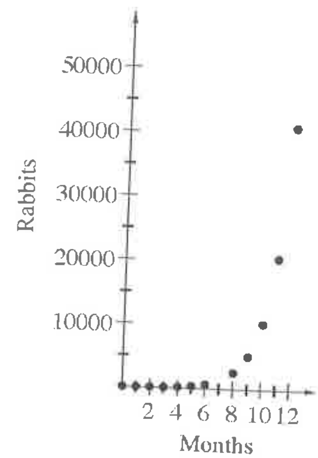
Case 4	
Months	Rabbits
0	2
1	8
2	32
3	128
4	512
5	2,048
6	8,192
7	32,768
8	131,072
9	524,288
10	2,097,152
11	8,338,608
12	33,552,432

5-4. A NEW FAMILY

Look back at the tables you created in problems 5-1 and 5-3.

a. What pattern do they all have in common? Functions that have this pattern are called **exponential functions**. **[They all change by multiplying by a constant.]**

b. Obtain the Lesson 5.1.1 Resource Page from your teacher. Graph the data for Case 2. Give a complete description of the graph. **[Graph shown at right. The graph from Case 2 is curved upward; the points are not connected because it is not possible to have fractions of bunnies; when x increases, y increases; the y-intercept is at 10; there is no x-intercept (students will study asymptotes in Chapter 7); the domain for the rabbits is from 0 months to infinity; the range for rabbits is from 10 rabbits to infinity; the only special point is the y-intercept.]**



5-5. LEARNING LOG

To represent the growth in number of rabbits in problems 5-1 and 5-3, you discovered a new function family that is not linear. Functions in this new family are called exponential functions. Throughout this chapter and later in Chapter 7, you will learn more about this special family of functions.

Write a Learning Log entry to record what you have learned so far about exponential functions. For example, what do their graphs look like? What patterns do you observe in their tables? Title this entry "Exponential Functions" and include today's date.

Review & Preview

5-6. What if the data for Lenny and George (from problem 5-1) matched the data in each table below? Assuming that the growth of the rabbits multiplies as it did in problem 5-1, complete each of the following tables. Show your thinking or give a brief explanation of how you know what the missing entries are.

a.

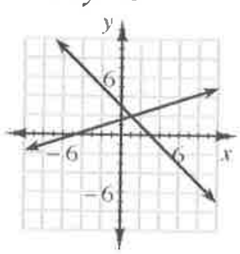
Months	Rabbits
0	4
1	12
2	36
3	**108**
4	**324**

b.

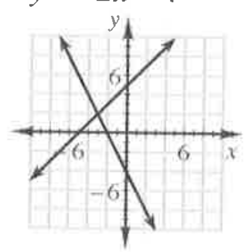
Months	Rabbits
0	6
1	**12**
2	24
3	**48**
4	96

5-7. Solve the following systems of equations algebraically. Then graph each system to confirm your solution. **[See graphs below.]**

a. $x+y=3$ **[(1, 2)]**
$x=3y-5$



b. $x-y=-5$ **[(−3, 2)]**
$y=-2x-4$



5-8. For the function $f(x)=\frac{6}{2x-3}$, find the value of each expression below.

a. $f(1)$ **[−6]**

b. $f(0)$ **[−2]**

c. $f(-3)$ **[$-\frac{2}{3}$]**

d. $f(1.5)$ **[undefined]**

e. What value of x would make $f(x)=4$? **[$x = 2.25$]**

5-9. Benjamin is taking Algebra 1 and is stuck on the problem shown below. Examine his work so far and help him by showing and explaining the remaining steps. **[$\frac{27b^3}{a^6}$]**

Original problem: Simplify $(3a^{-2}b)^3$.

He knows that $(3a^{-2}b)^3=(3a^{-2}b)(3a^{-2}b)(3a^{-2}b)$. Now what?

5-10. Simplify each expression below. Assume that the denominator in part (b) is not equal to zero.

a. $(x^3)(x^{-2})$ **[x]**

b. $\frac{y^5}{y^{-2}}$ **[y^7]**

c. 4^{-1} **[$\frac{1}{4}$]**

d. $(4x^2)^3$ **[$64x^6$]**

5-11. The equation of a line describes the relationship between the x- and y-coordinates of the points on the line.

a. Plot the points $(3, -1)$, $(3, 2)$, and $(3, 4)$ and draw the line that passes through them. State the coordinates of two more points on the line. Then answer this question: What will be true of the coordinates of any other point on this line? Now write an equation that says exactly the same thing. (Do not worry if it is very simple! If it accurately describes all the points on this line, it is correct.) **[Sample answers: (3, 0) and (3, 1); All points on this line have 3 as an x-coordinate. $x = 3$]**

b. Plot the points $(5, -1)$, $(1, -1)$, and $(-3, -1)$. What is the equation of the line that goes through these points? **[$y = -1$]**

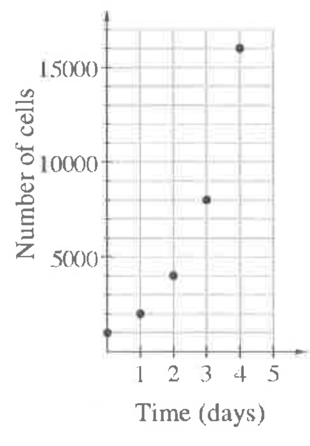
c. Choose any three points on the y-axis. What must be the equation of the line that goes through those points? **[$x = 0$]**

5-12. Jill is studying a strange bacterium. When she first looks at the bacteria, there are 1000 cells in her sample. The next day, there are 2000 cells. Intrigued, she comes back the next day to find that there are 4000 cells!

a. Should the graph of this situation be linear or curved? **[curved]**

b. Create a table and graph for this situation. The inputs are the days that have passed after she first began to study the sample, and the outputs are the number of cells of bacteria. **[See table and graph at right.]**

Time (days)	Number of cells
0	1000
1	2000
2	4000
3	8000
4	16,000



5-13. Write each expression below in a simpler form.

a. $\frac{5^{723}}{5^{721}}$ **[$5^2 = 25$]**

b. $\frac{3^{300}}{3^{249}}$ **[3^{51}]**

c. $\left(\frac{3 \cdot 4^3}{3^{-2} \cdot 4^{-7}}\right)^0$ **[1]**

d. $\left(\frac{4 \times 10^3}{10^{-2}}\right)^2$ **[1.6×10^{11}]**

5-14. Jackie and Alexandra were working on homework together when Jackie said, *"I got $x = 5$ as the solution, but it looks like you got something different. Which solution is right?"*

$$(x+4)^2 - 2x - 5 = (x-1)^2$$
$$x^2 + 16 - 2x - 5 = x^2 + 1$$
$$16 - 2x - 5 = 1$$
$$11 - 2x = 1$$
$$-2x = -10$$
$$x = 5$$

"I think you made a mistake," said Alexa. Did Jackie make a mistake? Help Jackie figure out whether she made a mistake and, if she did, explain her mistake and show her how to solve the equation correctly. Jackie's work is shown above right. **[Jackie squared the binomials incorrectly. It should be: $x^2 + 8x + 16 - 2x - 5 = x^2 - 2x + 1$, $6x + 11 = -2x + 1$, $8x = -10$, and $x = -1.25$.]**

5-15. Solve each of the following equations.

a. $\frac{m}{6} = \frac{15}{18}$ **[$m = 5$]**

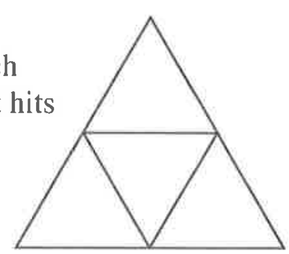
b. $\frac{\pi}{7} = \frac{a}{4}$ **[$a = \frac{4\pi}{7} \approx 1.80$]**

5-16. Write the equation of each line described below.

a. A line with slope –2 and y-intercept 7. **[$y = -2x + 7$]**

b. A line with slope $-\frac{3}{2}$ and x-intercept $(4, 0)$. **[$y = -\frac{3}{2}x + 6$]**

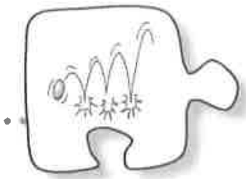
5-17. The dartboard shown at right is in the shape of an equilateral triangle. It has a smaller equilateral triangle in the center, which was made by joining the midpoints of the three edges. If a dart hits the board at random, what is the probability that:



a. The dart hits the center triangle? **[$\frac{1}{4}$]**

b. The dart misses the center triangle but hits the board? **[$\frac{3}{4}$]**

5.1.2 How high will it bounce?



Rebound Ratios

Many games depend on how a ball bounces. For example, if different basketballs rebounded differently, one basketball would bounce differently off of a backboard than another would, and this could cause basketball players to miss their shots. For this reason, manufacturers have to make balls' bounciness conform to specific standards. In this lesson, you will investigate the relationship between the height from which you drop a ball and the height to which it rebounds.

5-18. Listed below are "bounciness" standards for different kinds of balls.

- Tennis balls: Must rebound approximately 111 cm when dropped from 200 cm.
- Soccer balls: Must rebound approximately 120 cm when dropped from 200 cm onto a steel plate.
- Basketballs: Must rebound approximately 53.5 inches when dropped from 72 inches onto a wooden floor.
- Squash balls: Must rebound approximately 29.5 inches when dropped from 100 inches onto a steel plate at 70° F.

Discuss with your team how you can measure a ball's bounciness. Which ball listed above is the bounciest? Justify your answer. **[Teams should come to the idea of using this ratio: $\frac{\text{rebound height}}{\text{starting height}}$. The basketball is bounciest with a rebound ratio of 0.743.]**

5-19. THE BOUNCING BALL

How can you determine if a ball meets expected standards?

Your Task: With your team, find the rebound ratio for a ball. Your teacher will provide you with a ball and a measuring device. You will be using the same ball again later, so make sure you can identify which ball your team is using. Before you start your experiment, discuss the following questions with your team.

What do we need to measure?

How should we organize our data?

How can we be confident that our data is accurate?

You should choose one person in your team to be the recorder, one to be the ball dropper, and two to be the spotters. When you are confident that you have a good plan, ask your teacher to come to your team and approve your plan.

5-20. MODELING YOUR DATA

Work with your team to model the data you collected by considering parts (a) through (c) below.

a. In problem 5-19, does the height from which the ball is dropped depend on the rebound height, or is it the other way around? With your team, decide which is the independent variable and which is the dependent variable. **[Independent: starting height, Dependent: rebound height]**

b. Graph your results on a full sheet of graph paper. Draw a line that best fits your data. Should this line go through the origin? Why or why not? Justify your answer in terms of what the origin represents in the context of this problem. **[The data should be approximately linear. Yes; the rebound ratio is constant. The line should pass through the origin, because when the starting height is 0, the rebound height is 0.]**

c. Find an equation for your line.

5-21. What is the rebound ratio for your team's ball? How is the rebound ratio reflected in the graph of your line of best fit? Where is it reflected in the equation for your data? Where is it reflected in your table? **[Rebound ratios vary for different balls. The rebound ratio is the slope of the line of best fit. The rebound ratio is the coefficient of x in the equation of the line. In the table, the rebound ratio is $\frac{\Delta y}{\Delta x}$.]**

Save your data and your graph in a safe place. You will need them for the next lesson.

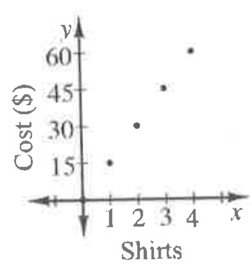
METHODS AND MEANINGS

MATH NOTES

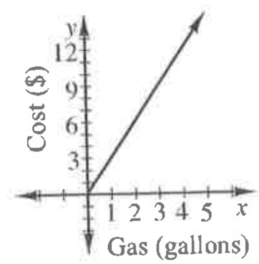
Continuous and Discrete Graphs

When the points on a graph are connected, and it *makes sense* to connect them, the graph is said to be **continuous**. If the graph is not continuous, and is just a sequence of separate points, the graph is called **discrete**. For example, the graph below left represents the cost of buying x shirts, and it is discrete because you can only buy whole numbers of shirts. The graph farthest right represents the cost of buying x gallons of gasoline, and it is continuous because you can buy any (non-negative) amount of gasoline.

Discrete Graph



Continuous Graph



Review & Preview

5-22. Solve each system of equations below.

a. $y = 3x + 1$ $x + 2y = -5$ **[(–1, –2)]**

b. $2x + 3y = 9$ $x - 2y = 1$ **[(3, 1)]**

5-23. Solve each equation for the indicated variable.

a. $t = an + b$ (for b) **[$b = t - an$]**

b. $\frac{y}{3} - a = b$ (for y) **[$y = 3(b + a)$]**

c. $m = \frac{y}{x}$ (for y) **[$y = mx$]**

d. $m = \frac{y}{x}$ (for x) **[$x = \frac{y}{m}$]**

5-24. Simplify each expression below.

a. $\frac{6x^2y^3}{3xy}$ **[$2xy^2$]**

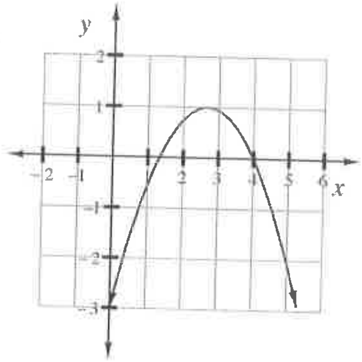
b. $(-mn)^3$ **[$-m^3n^3$]**

c. $(mn)^{-3}$ **[$\frac{1}{m^3n^3}$]**

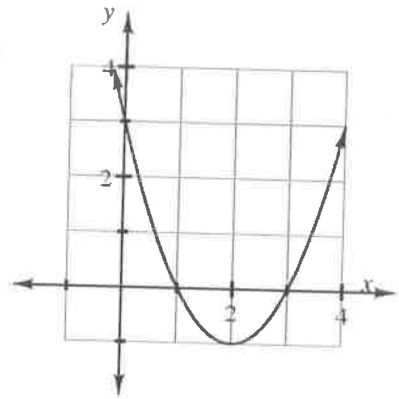
d. $\frac{3.2\times10^{-2}}{8\times10^{3}}$ **[4×10^{-6}]**

5-25. Determine the domain and range of each of the following graphs. **[a: domain: all numbers, range: $y \le 1$; b: domain: all numbers, range: $y \ge -1$; c: domain: all numbers, range: $y \le 0$; d: domain: all numbers, range: $y \ge -1$]**

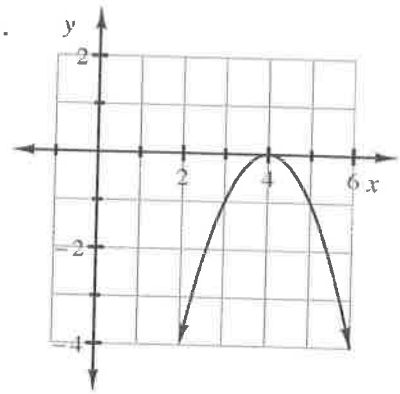
a.



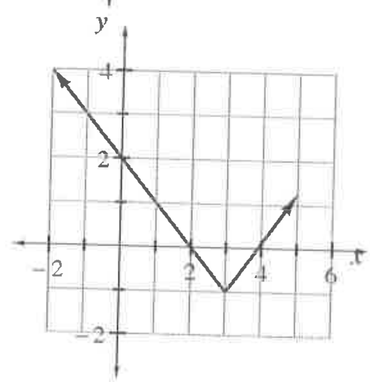
b.



c.

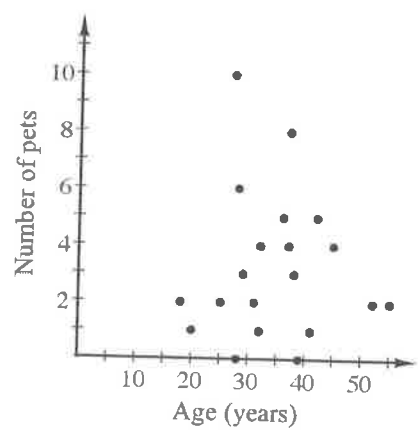


d.



5-26. The graph at right compares the age and the number of pets for a certain population.

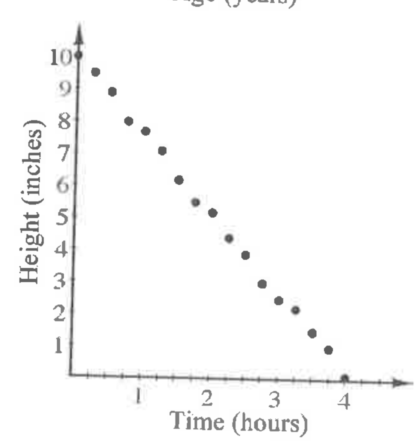
Describe the association for this population.
[There is no association between number of pets and age.]



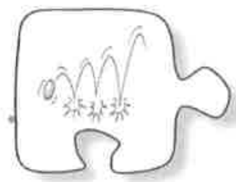
5-27. At an aunt's wedding, Nicolas collected data about an ice sculpture that was about to completely melt. A graph of his data is shown at right.

a. Calculate the equation of a line of best fit. **[$y = -\frac{5}{2}x + 10$]**

b. Based on your equation, how tall was the ice sculpture one hour before Nicolas started measuring? **[12.5 inches]**



5.1.3 What is the pattern?



The Bouncing Ball and Exponential Decay

In Lesson 5.1.2, you found that the relationship between the height from which a ball is dropped and its rebound height is determined by a constant multiplier. In this lesson, you will continue this investigation by exploring the mathematical relationship between how many times a ball has bounced and the height of each bounce.

5-28. Consider the work you did in Lesson 5.1.2, in which you found the rebound ratio for a ball.

a. What was the rebound ratio for the ball your team used? **[Answers vary.]**

b. Did the height you dropped the ball from affect this ratio? **[No.]**

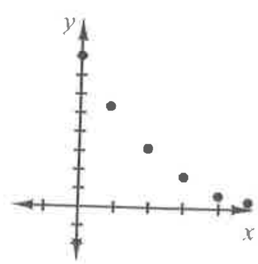
c. If you were to use the same ball again and drop it from *any* height, could you predict its rebound height? Explain how you would do this.
[Rebound ratio · drop height = rebound height.]

5-29. A MODEL FOR MANY BOUNCES

Imagine that you drop the ball you used in problem 5-19 from a height of 200 cm, but this time you let it bounce repeatedly.

a. As a team, discuss this situation. Then sketch a picture showing what this situation would look like. Your sketch should show a minimum of 6 bounces after you release the ball. **[Example graph shown below.]**

b. Predict your ball's rebound height after each successive bounce if its starting height is 200 cm. Create a table with these predicted heights. **[Answers will vary, but will be calculated by multiplying 200 by the rebound ratio for each bounce.]**

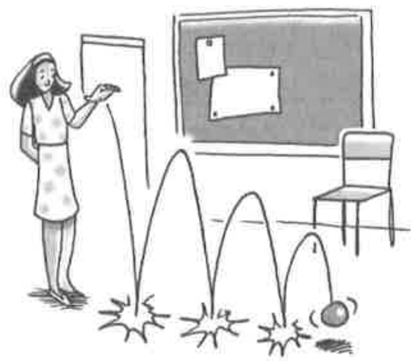


c. What are the independent and dependent variables in this situation? **[Independent: bounce number, dependent: height]**

d. Graph your predicted rebound heights. **[The graph should resemble the one at right.]**

e. Should the points on your graph be connected? How can you tell? **[This is a discrete situation (there is no 1.5th bounce), so the points should not be connected.]**

5-30. TESTING THE MANY-BOUNCE MODEL



Now you will test the accuracy of the predictions you made in problem 5-29.

Your Task: Test your predictions by collecting experimental data. Use the same team roles as you used in problem 5-19. Drop your ball, starting from an initial height of 200 cm, and record your data in a table. Then compare your experimental data to your predictions using your table and your graph. How do they compare? What might cause your experimental data to be different from your predictions? Do you think that your table and graph model the situation appropriately? Why or why not?

These suggestions will help you gather accurate data:

- Have a spotter catch the ball just as it reaches the top of its first rebound and have the spotter "freeze" the ball in place.
- Record the first rebound height and then drop the ball again from that new height.
- Catch and "freeze" it again at the second rebound height.
- Repeat this process until you have collected at least six data points (or until the height of the bounce is so small that it is not reasonable to continue).

5-31. Compare your graph for the height of successive bounces in problem 5-29 to the graph for drop height versus bounce height that you investigated in Lesson 5.1.2.

a. Can you use the same kind of equation to model the two situations? That is, what family of functions do you think would make the best fit for each data set? Discuss this with your team and be ready to report and justify your choice. **[The model for successive bounces is not linear. We cannot use a $y = mx + b$ equation to model the data. The linear family is not appropriate; the model cannot be quadratic, because the bounces never go back up again like a parabola does; the relationship must be exponential, though students might not realize that at this point since it is a decay rather than growth situation.]**

b. Describe how the pattern of growth for successive bounces is the same as or different from other models that you have looked at previously. **[The pattern of growth is that each successive bounce is the previous bounce multiplied by the rebound height which is similar to the bunny problem except that the multiplier is less than one.]**

5-32. If you continued to let your ball bounce uninterrupted, how high would the ball be after 12 bounces? Would the ball ever stop bouncing? Explain your answer in terms of both your experimental data and your equation. **[Answers vary depending on rebound ratio and would follow the equation $y = 200(r)^{12}$ though students are expected to do this calculation manually without the use of the equation. Of course in reality the ball would stop bouncing. The model, however, predicts that the ball would never stop bouncing.]**

5-33. Notice that your investigations of rebound patterns in Lessons 5.1.2 and 5.1.3 involved both a linear and an exponential model. Look back over your work and discuss with your team why each model was appropriate for its specific purpose. Be prepared to share your ideas with the class. **[Sample response: The height of a ball's rebound grows constantly as the drop height grows, so it makes sense that this would be a linear model. The height of each bounce is a constant multiple of its previous height, so it makes sense that, if left to bounce repeatedly, the ball's height would shrink exponentially.]**

Review & Preview

5-34. DeShawna and her team gathered data for their ball and recorded it in the table shown at right.

Drop Height	Rebound Height
150 cm	124 cm
70 cm	59 cm
120 cm	100 cm
100 cm	83 cm
110 cm	92 cm
40 cm	33 cm

a. What is the rebound ratio for their ball? **[Answers vary but should be close to 0.83.]**

b. Predict how high DeShawna's ball will rebound if it is dropped from 275 cm. Look at the precision of DeShawna's measurements in the table. Round your calculation to a reasonable number of decimal places. **[Approximately 228 cm. Since DeShawna measured to the nearest centimeter, a prediction rounded to the nearest centimeter would be reasonable.]**

c. Suppose the ball is dropped and you notice that its rebound height is 60 cm. From what height was the ball dropped? Use an appropriate precision for your answer. **[Approximately 72 cm.]**

d. Suppose the ball is dropped from a window 200 meters up the Empire State Building. What would you predict the rebound height to be after the first bounce? **[Approximately 166 meters.]**

e. How high would the ball in part (d) rebound after the second bounce? After the third bounce? **[Approximately 138 meters, approximately 114 meters.]**

5-35. Look back at the data given in problem 5-18 that describes the rebound ratio for an official tennis ball. Suppose you drop such a tennis ball from an initial height of 10 feet.

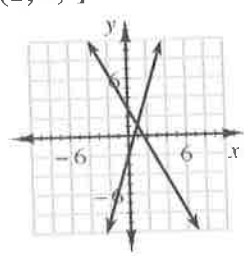
a. How high would it rebound after the first bounce? **[$10(0.555) = 5.55$ feet]**

b. How high would it rebound after the second bounce? **[$10(0.555)(0.555) = 3.08$ feet]**

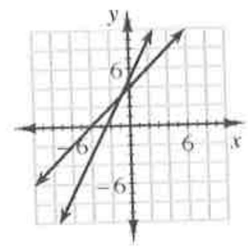
c. How high would it rebound after the fifth bounce? **[$10(0.555)^5 = 0.527$ feet]**

5-36. Solve the following systems of equations algebraically and then confirm your solutions by graphing. **[See graph below.]**

a. $y = 3x - 2$ **[(1, 1)]**
$4x + 2y = 6$



b. $x = y - 4$ **[(−1, 3)]**
$2x - y = -5$



5-37. Lona received a stamp collection from her grandmother. The collection is in a leather book and currently has 120 stamps. Lona joined a stamp club, which sends her 12 new stamps each month. The stamp book holds a maximum of 500 stamps.

a. Complete the table at right. **[144, 156, 168, 180]**

Month	Stamps
0	120
1	132
2	
3	
4	
5	

b. How many stamps will Lona have in one year from now? **[264 stamps.]**

c. Write an equation using function notation to represent the total number of stamps that Lona has in her collection after n months. Let the total be represented by $t(n)$. **[$t(n) = 12n + 120$]**

d. Solve your equation from part (c) for n when $t(n) = 500$. Will Lona be able to fill her book exactly with no stamps remaining? How do you know? When will the book be filled? **[$n = 31.67$; She will not be able to fill her book exactly, because 500 is not a multiple of 12 more than 120. The book will be filled after 32 months.]**

5-38. Use slope to determine whether the points $A(3,5)$, $B(-2,6)$, and $C(-5,7)$ are on the same line. Justify your conclusion algebraically. **[They are not on the same line; $m_{AB} = -\frac{1}{5}$, $m_{BC} = -\frac{1}{3}$, $m_{AC} = -\frac{1}{4}$.]**

5-39. Serena wanted to examine the graphs of the equations below on her graphing calculator. Rewrite each of the equations in ***y*-form** (when the equation is solved for *y*) so that she can enter them into the calculator.

a. $5-(y-2)=3x$
[$y=-3x+7$]

b. $5(x+y)=-2$
[$y=-x-\frac{2}{5}$]